IOPscience iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

Unusual solutions to the Yang-Baxter equation

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1987 J. Phys. A: Math. Gen. 20 1661

(http://iopscience.iop.org/0305-4470/20/7/013)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
IP Address: 129.252.86.83
The article was downloaded on 01/06/2010 at 05:28

Please note that terms and conditions apply.

J. Phys. A: Math. Gen. 20 (1987) 1661-1667. Printed in the UK

Unusual solutions to the Yang–Baxter equation

Ladislav Hlavatý

Institute of Physics, Czechoslovak Academy of Sciences, Na Slovance 2, 180 40 Prague 8, Czechoslovakia

Received 5 August 1986

Abstract. The complete list of non-symmetric eight-vertex constant solutions to the Yang-Baxter equation is obtained and new solutions dependent on more than one variable are presented. Corresponding spin Hamiltonians are derived by Sutherland's method.

1. Introduction

The Yang-Baxter equation (YBE)

$$R^{k_1k_2}_{j_1j_2}(u,v)R^{l_1k_3}_{k_1j_3}(u,w)R^{l_2l_3}_{k_2k_3}(v,w)=R^{k_2k_3}_{j_2j_3}(v,w)R^{k_1l_3}_{j_1k_3}(u,w)R^{l_1l_2}_{k_1k_2}(u,v) \tag{1.1}$$

where $j_1, j_2, j_3, k_1, k_2, k_3, l_1, l_2, l_3=1,2,\ldots,N$, as well as its classical counterpart emerge in many branches of theoretical physics (for a review see, e.g., [1] and the most recent application in [2]).

Up to now many solutions of (1.1) have been found (see [3-7] and references therein). Most of them are functions of only one variable $R(u,v)=R(u-v)$ and they satisfy

$$R^{cd}_{ab}(u=0)=P^{cd}_{ab}:=\delta^d_a\delta^c_b \tag{1.2}$$

so that one can ask if (1.2) does not hold for all the solutions of (1.1). The answer is negative and solutions that do not satisfy (1.2) are presented in this paper. We shall investigate the simplest case $N=2$ and assume that the matrices R have the non-symmetric eight-vertex form

$$R^{cd}_{ab}(u,v)=\begin{pmatrix} a & 0 & 0 & \tilde{d} \\ 0 & b & \tilde{c} & 0 \\ 0 & c & \tilde{b} & 0 \\ d & 0 & 0 & \tilde{a}\end{pmatrix}. \tag{1.3}$$

Solutions of the form (1.3) where the entries depend only on the difference $u-v$ were classified in [6] but, as it will be seen below, not completely.

In this paper we admit the entries being genuine functions of two variables, i.e. not necessarily of their difference. Let us recall that the YBE can be considered as a consequence of the 'commutation relation' [1]

$$\hat{R}(u,v)[L(u)\otimes L(v)]=[L(v)\otimes L(u)]\hat{R}(u,v) \tag{1.4}$$

where the matrix $\hat{R}$ is simply related to R and there is no *a priori* reason that R be a function of the difference $u-v$.

0305-4470/87/071661+07\$02.50 © 1987 IOP Publishing Ltd

Moreover, we shall assume that u and v can be multicomponent objects, i.e. u, $v \in \mathbb{C}^n$ in general.

An example of such a solution is the free-fermion one [8, 13] where

$$\begin{aligned}
a(u, v) &= 1 - u_1 v_1 e(t) \\
\tilde{a}(u, v) &= e(t) - u_1 v_1 \\
b(u, v) &= u_1 - v_1 e(t) \\
\tilde{b}(u, v) &= v_1 - u_1 e(t) \\
c(u, v) &= \tilde{c}(u, v) = (\mathrm{i}/2)(1 - u_1^2)(1 - v_1^2)(1 - e(t))/\mathrm{sn}(t/2, k) \\
d(u, v) &= \tilde{d}(u, v) = (k/2)(1 - u_1^2)(1 - v_1^2)(1 + e(t))\,\mathrm{sn}(t/2, k)
\end{aligned} \tag{1.5}$$

where

$$t = u_2 - v_2 \qquad e(t) = \mathrm{cn}(t, k) + \mathrm{i}\,\mathrm{sn}(t, k) \tag{1.6}$$

and sn, cn are Jacobi's elliptic functions.

Another example is the non-symmetric six-vertex solution

$$\begin{aligned}
a(u, v) &= u_1 v_1^{-1} \sin(u_2 - v_2 + k) \\
\tilde{a}(u, v) &= u_1^{-1} v_1 \sin(u_2 - v_2 + k) \\
b(u, v) &= u_1^{-1} v_1^{-1} \sin(u_2 - v_2) \\
\tilde{b}(u, v) &= u_1 v_1 \sin(u_2 - v_2) \\
c(u, v) &= \tilde{c}(u, v) = \sin k \qquad d(u, v) = \tilde{d}(u, v) = 0
\end{aligned} \tag{1.7}$$

that corresponds to the ferroelectrics in an electric field or XXZ spin model in a magnetic field [9].

Solutions presented in this paper are classified from the point of view of their values in $u = v$.

2. Constant solutions

The first problem we are going to solve is: which constant matrices can stand on the right-hand side of (1.2)?

They must satisfy (1.1) with $u = v = w$ so that we must look for solutions of the 'constant YBE'

$$R^{k_1 k_2}_{j_1 j_2} R^{l_1 k_3}_{k_1 j_3} R^{l_2 l_3}_{k_2 k_3} = R^{k_2 k_3}_{j_2 j_3} R^{k_1 l_3}_{j_1 k_3} R^{l_1 l_2}_{k_1 k_2}. \tag{2.1}$$

The problem of finding all solutions of (2.1) for arbitrary N was posed in [3] and, to the best of my knowledge, has not yet been solved. A construction of the so-called quasiclassical constant solutions to (2.1), provided solutions of the classical YBE are given, was presented in [10].

The solution space of equation (2.1) is invariant under the transformations

$$R = k(T \otimes T) R (T \otimes T)^{-1} \tag{2.2}$$

where $T \in \mathrm{GL}(2, C)$, k is a constant $\neq 0$. We can exploit this symmetry with T (anti)diagonal to get $d = \tilde{d}$ or $\tilde{d} = 0$ and then insert (1.3) into (2.1).

We obtain a system of quadratic equations for the entries of (1.3) (cf [6]). The system can be easily solved and the following list of solutions results:

$$R_0=\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}\qquad R_1=\begin{pmatrix}1&0&0&i\\0&1&1&0\\0&1&-1&0\\i&0&0&1\end{pmatrix}$$

$$R_2=\begin{pmatrix}1+t&0&0&1\\0&(1+t^2)^{1/2}&1&0\\0&1&(1+t^2)^{1/2}&0\\1&0&0&1-t\end{pmatrix}\qquad R_3=\begin{pmatrix}1&0&0&0\\0&\varepsilon&0&0\\0&1-t&\varepsilon t&0\\1&0&0&-t\end{pmatrix}$$

$$R_4=\begin{pmatrix}1&0&0&0\\0&\varepsilon t&1-t&0\\0&0&\varepsilon&0\\1&0&0&-t\end{pmatrix}\qquad R_5=\begin{pmatrix}1&0&0&0\\0&r&0&0\\0&1-rt&t&0\\0&0&0&1\end{pmatrix}$$

$$R_6=\begin{pmatrix}1&0&0&0\\0&r&0&0\\0&1-rt&t&0\\0&0&0&-rt\end{pmatrix}\qquad R_7=\begin{pmatrix}1&0&0&0\\0&\varepsilon&0&0\\0&0&\varepsilon&0\\1&0&0&\varepsilon'\end{pmatrix}$$

$$R_8=\begin{pmatrix}1&0&0&0\\0&r&0&0\\0&0&s&0\\0&0&0&t\end{pmatrix}\qquad R_9=\begin{pmatrix}0&0&0&1\\0&0&t&0\\0&t&0&0\\1&0&0&0\end{pmatrix}$$

where ε^2, $\varepsilon'^2=1$ and r, s, t are arbitrary constants different from zero. By their construction the matrices form the complete list of solutions to (2.1) of the form (1.3) up to the transformation (2.2). (Only regular matrices are considered.)

Note that the matrices R_3–R_7 have non-symmetric antidiagonals. Similar (but non-constant) solutions, the so-called seven-vertex models, were presented in [6]. Here we can see that there are also five-vertex and non-symmetric six-vertex solutions. Their non-constant versions will be constructed below.

3. Non-constant solutions

Solutions mentioned in the introduction as well as those presented in [6] satisfy $R(u,u)=R_0=P$. Here we shall concentrate on solutions satisfying

$$R(u,u)=R_i \qquad i\in\{1,2,\ldots,9\}. \tag{3.1}$$

We can exploit several facts: first, if the solution satisfies (3.1) then the equation (1.1) with $u=v$ provides a system of purely algebraic (i.e. not functional-algebraic) equations that further restricts the ansatz (1.3). (It is an identity for $R(u,u)=R_0$.)

Second, the solution space of equation (1.1) is invariant under the transformations

$$R'(u,v)=k(u,v)[T(u)\otimes T(v)]R(u,v)[T(u)\otimes T(v)]^{-1} \tag{3.2}$$

where the arbitrary functions $k(u, v)$ and $T(u)$ are scalar and $GL(2, C)$-valued, respectively. Another useful symmetry of (1.1) is

$$R'(u, v) = PR(v, u)P \tag{3.3}$$

where P is given by (1.2). These symmetries can be exploited to achieve $\{d(u, v) = \tilde{d}(u, v)$ or $\tilde{d}(u, v) = 0\}$ and $\{c(u, v) = \tilde{c}(u, v)$ or $\tilde{c}(u, v) = 0\}$ so that the solutions can be classified according to the number of zeros on the antidiagonal of (1.3).

Finally, there are trivial solutions of (1.1) of the form

$$R(u, v) = f(u, v)[\sigma_i \otimes \sigma_i] + g(u, v)[\sigma_j \otimes \sigma_j] \qquad \text{(no sums)} \tag{3.4}$$

where $i, j \in \{1, 2, 3, 4\}$, σ_i are Pauli matrices and unit matrix. Another trivial solution is an arbitrary diagonal matrix.

There are only trivial solutions of (1.1) satisfying (3.1) for $i = 8, 9$ and one can show that there is no non-constant (up to a scalar factor) solution satisfying $R(u, u) = R_7$.

The solutions satisfying (3.1) for $i = 1, 2, \ldots, 6$ up to the transformation (2.2) are displayed in table 1.

Table 1. Entries of the solutions $R(u, v)$ satisfying (3.1). $D^2 = \mathrm{i}(1 - u^2)(1 - v^2)/2$, $k =$ constant, $\varepsilon^2 = 1$.

Solution \ Entry	a	$\tilde{a}$	b	$\tilde{b}$
I	u/v	u/v	u/v	$-u/v$
II	$\mathrm{i} - uv$	$1 - \mathrm{i}uv$	$v - \mathrm{i}u$	$u - \mathrm{i}v$
III	1	$-v$	ε	εv
IV	1	$-u$	εu	ε
V	u/v	v/u	$(uv)^{-1}$	kuv
VI	u_1/v_2	v_2/u_1	$(u_1 v_2)^{-1}$	$-u_1 v_2$
Solution \ Entry	c	$\tilde{c}$	d	$\tilde{d}$
I	1	1	i	i
II	D	D	D	D
III	$1 - v$	0	$1 + v$	0
IV	0	$1 - u$	$1 + u$	0
V	$1 - k$	0	0	0
VI	$v_1 v_2^{-1} + v_1^{-1} v_2$	0	0	0

4. Spin Hamiltonians

There is a well known connection between the solutions of the YBE and quantum Hamiltonians

$$H = \sum_{i=1}^{N} H_{i,i+1} = \sum_{i=1}^{N} h_{\alpha\beta} \sigma_i^\alpha \sigma_{i+1}^\beta. \tag{4.1}$$

If $R(u, v)$ is a solution to (1.1) then

$$L^{b\beta}_{a\alpha}(v) := R^{\beta b}_{\alpha a}(u, v) \tag{4.2}$$

and

$$\hat{R}^{cd}_{ab}(u, v) := R^{dc}_{ab}(v, u) \tag{4.3}$$

satisfy the commutation relation (1.4) and the matrix L can be used for the construction of transfer matrices commuting with the spin Hamiltonian (4.1).

For solutions satisfying $R(u, u) = R_0 = P$ the Hamiltonian is given by Baxter's formula [11]. However, there is no such formula for solutions satisfying (3.1) and we have to use Sutherland's method for construction of the Hamiltonian.

The sufficient condition for commutation of the transfer matrix (for details of the notation see [12])

$$T(u) = \mathrm{Tr} \prod_{i=1}^{N} L_i(u) \tag{4.4}$$

and the Hamiltonian of the form (4.1) is the existence of a matrix Q such that

$$[L_i L_{i+1}, H_{i,i+1}] = L_i Q_{i+1} - Q_i L_{i+1}. \tag{4.5}$$

For

$$L = \begin{pmatrix} a & 0 & 0 & \tilde{d} \\ 0 & \tilde{b} & c & 0 \\ 0 & \tilde{c} & b & 0 \\ d & 0 & 0 & \tilde{a} \end{pmatrix} \tag{4.6}$$

we can assume that the matrix Q is also of the eight-vertex form and the Hamiltonian has the form

$$H = \sum_{i=1}^{N} [J_x \sigma_i^x \sigma_{i+1}^x + J_y \sigma_i^y \sigma_{i+1}^y + J_z \sigma_i^z \sigma_{i+1}^z + h\sigma_i^z$$
$$+ M(\sigma_i^x \sigma_{i+1}^y - \sigma_i^y \sigma_{i+1}^x) + K(\sigma_i^x \sigma_{i+1}^y + \sigma_i^y \sigma_{i+1}^x)]. \tag{4.7}$$

From Sutherland's equation (4.5) we then get the following relations between the elements of the matrix L and the coefficients of the Hamiltonian

$$d[(a\tilde{b} - b\tilde{a})J_z + \mathrm{i}FM] = 0 \tag{4.8}$$

$$c[(a\tilde{b} - b\tilde{a})J_z - \mathrm{i}FM] = 0 \tag{4.9}$$

$$d[(a\tilde{b} + b\tilde{a})J_z - (F + 2c\tilde{c})J] + c(ab + \tilde{a}\tilde{b})(\Gamma - \mathrm{i}K) = 0 \tag{4.10}$$

$$c[a\tilde{b} + b\tilde{a})J_z - (F + 2d\tilde{d})J] + \tilde{d}(ab + \tilde{a}\tilde{b})(\Gamma + \mathrm{i}K) = 0 \tag{4.11}$$

$$(\Gamma - \mathrm{i}K)(a^2 + b^2 - \tilde{a}^2 - \tilde{b}^2) = 4\tilde{c}\tilde{d}h \tag{4.12}$$

where

$$J_x = J + \Gamma \qquad J_y = J - \Gamma$$

and

$$F = a\tilde{a} + b\tilde{b} - c\tilde{c} - d\tilde{d} \tag{4.13}$$

plus the equations obtained from (4.8)-(4.12) by the transformation $[a, b, c, d \leftrightarrow \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d};\ K \to -K,\ M \to -M]$.

For the solutions from table 1 we get the following values of coefficients of the spin Hamiltonian (4.7).

Solution I: $J_z = J = h = K = 0$ and Γ, M = arbitrary.

Solution II: $J_z = K = 0$, $J:\Gamma:h = 2u:-\mathrm{i}(1-u^2):(1+u^2)$ and M = arbitrary.

Solution III: $J_z = 0$, $\Gamma = h = \mathrm{i}K$, and J, M = arbitrary.

Solution IV: $J_z = 0$ and J, Γ, h, K, M = arbitrary.

Solution V: $K = \Gamma = 0$, $J_z = (u^2/2)(1+k)(J+\mathrm{i}M)$, and J, h, M = arbitrary.

Solution VI: $J_z = K = \Gamma = 0$ and J, h, M = arbitrary.

Here u and k are arbitrary constants. They are the parameters of the matrix L (see (4.6) and (4.2)).

5. Conclusions

We have found several solutions of the YBE depending in general on more than one variable. They are of the form (1.3) with the entries given in table 1. All but solution V are the free-fermion types. Each of the solutions represents a class of solutions obtained from a given one by the transformation (3.2) and by the transformation $u' = f(u)$, $v' = f(v)$ where f is an arbitrary function.

The first two solutions are genuine eight-vertex ones. Solution I can be expressed as a function of the difference of variables and, therefore, it should have been included in the classification [6]. Moreover, it is a free-fermion type solution that cannot be parametrised in the Felderhof way [13] (cf [4]). Solution II can be obtained as the limit $k = 1$, $t \to \infty$ of the solution (1.5).

The other solutions are of the five-vertex and non-symmetric six-vertex types. Together with the constant solution R_7 they represent all possible solutions of their classes, i.e. cases (1.3) with $d = \tilde{d} = \tilde{c} = 0$, $\tilde{d} = \tilde{c} = 0$, $\tilde{d} = c = 0$, $\tilde{d} = \tilde{c} = c = 0$. Note that solutions III and IV depend on one variable only but not on the difference of variables as usually.

The spin Hamiltonians corresponding to the newly found solutions are generalised XXZ or XY models. Unfortunately, some of them are non-Hermitian.

Acknowledgments

Discussion of the presented solutions with N Yu Reshetikhin and E K Sklyanin is gratefully acknowledged.

References

[1] Kulish P P and Sklyanin E K 1982 *Lecture Notes in Physics* **151** (Berlin: Springer)
[2] Ward R 1985 *Phys. Lett.* **112A** 3
[3] Kulish P P and Sklyanin E K 1980 *Zap. Nauch. Sem. LOMI* **95** 95; 1982 *J. Sov. Math.* **19** 1596
[4] Krichever I M 1981 *Funkt. Anal. Pril.* **15** 22
[5] Perk J H H and Schultz C L 1981 *Phys. Lett.* **84A** 407
[6] Sogo K, Uchinami M, Akutsu Y and Wadati M 1982 *Prog. Theor. Phys.* **68** 508
[7] Jimbo M and Miwa T 1985 *Nucl. Phys.* B **257** 1
[8] Bazhanov V V and Stroganov Yu G 1985 *Teor. Mat. Fis.* **62** 377
[9] Hlavaty L 1985 *Dubna preprint* JINR E5-85-959

[10] Drinfeld V G 1983 *Dokl. Akad. Nauk SSSR* **273** 531
[11] Baxter R J 1977 *Phys. Rev. Lett.* **26** 834
[12] Sutherland B 1970 *J. Math. Phys.* **11** 3183
[13] Felderhof B U 1973 *Physica* **66** 279